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Liquidity Trap and Optimal Monetary Policy: Evaluations for U.S. Monetary Policy*

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Abstract

This paper shows that the Fed's exit strategy works as optimal monetary policy in a liquidity trap. We use the conventional new Keynesian model including a recent inflation persistence and confirm several similarities between optimal monetary policy and the Fed's monetary policy. The zero interest rate policy continues even after inflation rates are sufficiently accelerated over the 2 percent target and hit a peak. Under optimal monetary policy, the zero interest rate policy continues until the second quarter of 2022 and the Fed terminates it one quarter earlier. Eventually, inflation rates exceed the target rate for over three years until the latest quarter. The policy rates continue to overshoot the long-run level to suppress high inflation rates.

Furthermore, high inflation rates under optimal monetary policy can explain about 70 percent of the inflation data for 2021 and 2022 years. However, these are still lower than the inflation data. This is because optimal monetary policy raises the policy rates faster than the Fed does. The remaining 30 percent of inflation rates can be constrained by the Fed's more aggressive monetary policy tightening after the zero interest rate policy.

JEL Classification: E31; E52; E58; E61
Keywords: liquidity trap; optimal monetary policy; inflation persistence; forward guidance

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1 Introduction

The theory of monetary policy has been developed since the 1990s based on a new Keynesian model as represented by Clarida et al. (1999) and Woodford (2003). Woodford (2003) finds history dependence as a general property of optimal monetary policy with commitment in a purely forward-looking new Keynesian model. He shows that the forward-looking economy and history dependence are two sides of a coin in optimal monetary policy. Eggertsson and Woodford (2003b,a), Jung et al. (2001, 2005), and Adam and Billi (2006) extend optimal monetary policy analysis with commitment to an economy in a liquidity trap and show that a robust conclusion about a feature of optimal monetary policy is history dependence. The consequence of optimal monetary policy under commitment in a liquidity trap is predicted by these papers. However, such predictions have not been evaluated in the past two decades. Now, we show the answer.

During the pandemic period, the FOMC lowered the target range for the federal funds rate to 0 to 1/4 percent on March 15, 2020. At the same time, the FOMC has declared a clear commitment to stimulate the economy and has kept its promise. The Fed commits that "The Committee expects to maintain an accommodative stance of monetary policy until labor market conditions have reached levels consistent with the Committee's assessments of maximum employment and inflation has risen to 2 percent and is on track to moderately exceed 2 percent for some time (9/16/2020)" by the Federal Open Market Committee statements. Furthermore, the Fed states "With inflation having exceeded 2 percent for some time, the Committee expects it will be appropriate to maintain this target range until labor market conditions have reached levels consistent with the Committee's assessments of maximum employment (12/15/2021)." In these statements, the Fed declares to allow inflation exceeding the 2 percent target to stimulate an economy and to escape from deflation. This inflation overshooting is consistent with optimal monetary policy with commitment. In reality, the Fed terminated the zero interest rate policy in March 2022 after inflation rates sufficiently exceeded the 2 percent target. Inflation rates had hit the peak before the monetary policy tightening and then are slowly declining.

In this paper, we show optimal monetary policy in a liquidity trap using a hybrid new Keynesian model including inflation persistence. First, we analytically derive optimal monetary policy and investigate its features. The novel feature is that optimal monetary policy changes with the degree of inflation persistence. A central bank should implement both history dependent and forward-looking responses to inflation rates and the output gap. Second, we show normative analyses and examine numerical simulations of when to exit from the zero interest rate policy. The optimal timing of ending the zero interest rate policy becomes relatively earlier as inflation persistence becomes larger. In the case of a higher degree of inflation persistence and a larger shock, the zero interest rate policy can be terminated even while the natural rate shock does not disappear and is below zero, that is, monetary tightening is front-loaded. We also observe such a front-loaded tightening against the peak inflation rate. The optimal exit policy from a liquidity trap drastically changes depending on inflation persistence and the size of shocks, despite previous studies stressing a history-dependent monetary policy.

Then, we apply our model to U.S. monetary policy and show that optimal commitment monetary policy replicates U.S. monetary policy and economy during and after the pandemic period. In particular, optimal commitment policy can explain about 70 percent of the inflation surge for 2021 and 2022 years. Even when we assume alternative scenarios for the natural interest rates and weakened forward guidance, our conclusion that optimal commitment policy well explains the inflation surge does not change. Our simulation suggests that a slower interest rate hike after the zero interest rate policy by the Fed induces recent excessive high inflation rates. The remaining 30 percent of inflation rates can be constrained by the Fed's more aggressive monetary tightening. It implies that the exit strategy from a liquidity trap depends on how quickly monetary policy tightens and when the zero interest rate policy ends.

Our paper is related to three strands of previous literature. First, our paper is related to optimal monetary policy in the model with inflation persistence such as in Woodford (2003) and Steinsson (2003). In particular, Woodford (2003) derives the Phillips curve including inflation inertia by the indexation rule. Our paper differs from these two papers in that we consider the zero lower bound on nominal interest rates. Terminating the zero interest rate policy depends on the extensive progress of inflation rates through backward-looking adjustment by indexation.

Second, our paper is related to optimal monetary policy in a liquidity trap. Eggertsson and Woodford (2003b,a) and Jung et al. (2001, 2005) show that the optimal commitment policy is history dependent so that a central bank continues a zero interest rate policy even after the natural rate turns positive.[1] Adam and Billi (2006, 2007) and Nakov (2008) solve the optimal commitment policy as well as the discretionary policy under the zero lower bound on nominal interest rates with stochastic shocks. Werning (2011) shows that the future consumption boom as well as the future high inflation play important roles in mitigating a liquidity trap. Evans et al. (2015) show an exit strategy from the zero interest rate policy under a suboptimal policy, i.e., optimal discretionary policy, using a purely forward-looking model and a purely backward-looking model. As an independent work for a deterministic shock, Michau (2019) shows optimal monetary and fiscal policy in a liquidity trap for a hybrid new Keynesian model and concludes that inflation persistence requires an early monetary tightening.[2] All these papers are the foundation for our paper and we evaluate real U.S. monetary policy after the last pandemic.

Third, our paper is also related to former empirical papers. Empirical studies using U.S. economic data show that the inflation rate is highly persistent and the Phillips curve is both forward-looking and backward-looking. Fuhrer and Moore (1995) and Galí and Gertler (1999) show that a hybrid Phillips curve rather than a purely forward-looking Phillips curve is suitable for monetary policy analyses. Christiano et al. (2005) and

[1]Eggertsson and Woodford (2006) and Eggertsson (2006, 2008, 2012) reveal roles of fiscal policy as well as monetary policy in a liquidity trap.

[2]There are many other influential papers regarding optimal monetary policy in a liquidity trap. For example, Jeanne and Svensson (2007) show the important role of currency depreciation and price level targeting as a commitment device to escape from a liquidity trap. Billi (2011) focuses on the optimal long-run inflation rate to preempt falling into a liquidity trap. Fujiwara et al. (2013) extend the model to the open economy and show an optimal zero interest rate policy in a global liquidity trap.

Smets and Wouters (2007) estimate the hybrid Phillips curve in a dynamic stochastic general equilibrium model and it suits the U.S. economy. Benati (2008) estimates sticky-price DSGE models with hybrid Phillips curves for several countries. For the U.S., on the sample from 1947Q1 to 2005Q4, estimations show strong evidence of high structural inflation persistence. For data after the Volkers stabilization of 1983Q1, however, an estimated indexation becomes lower. Moreover, Cogley and Sbordone (2008) show that the inflation rate is explained by purely forward-looking behavior for post-WWII data after controlling a trend shift in an inflation rate. Carvalho et al. (2019) show a small indexation parameter using data 1955Q1-2015Q4 for the U.S.[3] For the last few years, Kiley (2023) focuses on the evolution of the inflation persistence in a Phillips Curve using the Bayesian approach. The paper shows that inflation persistence drastically increases by post-2019 experience and is estimated at 0.86. It requires us to set a highly persistent indexation parameter as one in our model. These findings suggest that the degree of indexation has changed over time in the U.S.

The remainder of the paper proceeds as follows. Section 2 presents a model of the economy with inflation persistence. Section 3 derives an optimal monetary policy in a liquidity trap and Section 4 examines numerical simulations to show the optimal exit strategy from a zero interest rate policy to the natural rate shocks. Section 5 shows that a negative cost-push shock induces deflation under optimal commitment policy. We evaluate recent U.S. monetary policy in Section 6. Section 7 concludes.

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[3]Benati (2008) estimates an indexation parameter γ for hybrid new Keynesian Phillips curve in Section 3 on the sample from 1947Q1 to 2005Q4 as 0.908 and for the sample after the Volkers stabilization of 1983Q1 as 0.619. Carvalho et al. (2019) show $\gamma = 0.128$ using data 1955Q1-2015Q4 for the U.S.

2 The Model

We use a new Keynesian model proposed by Woodford (2003). The macroeconomic structure is expressed by the following three equations:

$$x_t = \mathrm{E}_t x_{t+1} - \chi \left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n\right), \tag{1}$$

$$\pi_t - \gamma \pi_{t-1} = \kappa x_t + \beta \left(\mathrm{E}_t \pi_{t+1} - \gamma \pi_t\right) + \mu_t, \tag{2}$$

$$r_t^n = \rho_r r_{t-1}^n + \epsilon_t^r, \tag{3}$$

where χ, κ, β, γ, and ρ_r are parameters, satisfying $\chi > 0$, $\kappa > 0$, $0 < \beta < 1$, $0 \leq \gamma \leq 1$, and $0 \leq \rho_r < 1$. x_t, i_t and π_t denote the output gap, the nominal interest rate (or policy rate), and the rate of inflation in period t, respectively. The expectations operator E_t covers information available in period t. r_t^n is the natural rate of interest, which is assumed to follow an AR(1) process. ϵ_t^r is i.i.d. disturbance with variances of σ_r. μ_t is the cost-push shock that is i.i.d. disturbance with variances of σ_μ.

Equation (1) is the forward-looking IS curve. The IS curve states that the current output gap is determined by the expected value of the output gap and the deviation of the current real interest rate, defined as $i_t - \mathrm{E}_t \pi_{t+1}$, from the natural rate of interest.

Equation (2) is the hybrid Phillips curve. γ denotes the degree of inflation persistence. In particular, when $\gamma = 0$, the hybrid Phillips curve collapses to a purely forward-looking Phillips curve, in which current inflation depends on expected inflation and the current output gap. When $0 < \gamma \leq 1$, the Phillips curve is both forward-looking and backward-looking and the current inflation rate depends on the lagged inflation rate as well as the expected inflation and the current output gap. As γ approaches one, the coefficient on the lagged inflation rate approaches 0.5.

In this paper, we assume inflation persistence with indexation. Specifically, we follow Woodford (2003), which derives the Phillips curve including inflation inertia with a micro-foundation.[4] In the indexation rule, some firms that cannot reoptimize their own goods

[4]There are several theoretical foundations to introduce inflation persistence. For example, Mankiw and Reis (2002) introduce information rigidity to produce inflation persistence. Milani (2007) points out the importance of an agent's learning for inflation persistence.

prices adjust current prices based on the past inflation rate. The indexation mechanism is empirically supported by Christiano et al. (2005) and Smets and Wouters (2007). We can analyse both the purely forward-looking Phillips curve and the hybrid Phillips curve by changing the parameters of inflation persistence.

Next, we consider the central bank's intertemporal optimization problem. The central bank sets the nominal interest rate i_t so as to minimize the approximated welfare loss $\mathcal{L}_t$ defined as

$$\mathcal{L}_t = \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \mathrm{L}_{t+i}, \tag{4}$$

where L_t is the period loss function obtained by second-order approximation of the household utility function. In an economy with inflation inertia, Woodford (2003) shows that L_t is given by

$$\mathrm{L}_t = (\pi_t - \gamma\pi_{t-1})^2 + \lambda_x x_t^2, \tag{5}$$

where λ_x is a non-negative parameter. A central bank needs to stabilize $\pi_t - \gamma\pi_{t-1}$ in approximation rather than the inflation rate itself when inflation exhibits intrinsic persistence. In an economy without inflation persistence, dispersion comes from an environment where some firms reoptimize prices and other firms do not change prices at all. In an economy with indexation on inflation rates, however, dispersion comes from an environment where some firms not reoptimizing their prices follow the past inflation rate with a certain degree γ and other firms reoptimize prices. Therefore, to minimize price dispersion, a central bank needs to set the current inflation rate so as to be close to the adjusted lagged inflation rate. This is eventually consistent with the Fed's forward guidance to allow inflation rates to flexibly exceed a target level of inflation rate. The loss function induces a looser commitment to anchor inflation rates to a target level than a loss function to minimize a deviation of the inflation rate itself from the target. However, it should be noted that we show optimal monetary policy that maximizes the household's utility regardless of the approximation.

Finally, we impose a nonnegativity constraint on the nominal interest rate:

$$i_t \geq 0. \tag{6}$$

It should be noted that the presence of a nonnegativity constraint introduces nonlinearity in an otherwise linear-quadratic model. The central bank maximizes equation (4) subject to equations (1)-(3) and (6).

3 Optimal Monetary Policy in a Liquidity Trap

We analytically characterize optimal monetary policy in a liquidity trap and clarify the implication of an optimal exit strategy. Optimal monetary policy under the zero lower bound on the nominal interest rate in a timeless perspective is expressed by the solution of the optimization problem.[5] To investigate features of optimal monetary policy, we denote the degree of inflation persistence in the hybrid Phillips curve as γ_{pc} and that in the period loss function as γ_{loss}. This setup is just to clarify the mechanism of inflation persistence and we set $\gamma_{pc} = \gamma_{loss} = \gamma$ in the benchmark. The optimization problem is represented by the following Lagrangian form:

$$\mathcal{L} = \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left\{ \begin{array}{c} \left(\pi_{t+i} - \gamma_{loss}\pi_{t+i-1}\right)^2 + \lambda_x x_{t+i}^2 \\ -2\phi_{1t+i}\left[x_{t+i+1} - \chi\left(i_{t+i} - \pi_{t+i+1} - r_{t+i}^n\right) - x_{t+i}\right] \\ -2\phi_{2t+i}\left[\kappa x_{t+i} + \beta\left(\pi_{t+i+1} - \gamma_{pc}\pi_{t+i}\right) - \pi_{t+i} + \gamma_{pc}\pi_{t+i-1}\right] \end{array} \right\},$$

where ϕ_1 and ϕ_2 are the Lagrange multipliers associated with the IS constraint and the Phillips curve constraint, respectively. We differentiate the Lagrangian with respect to π_t, x_t, and i_t under the nonnegativity constraint on nominal interest rates to obtain the first-order conditions:

$$-\beta\gamma_{loss}\left(\mathrm{E}_t\pi_{t+1} - \gamma_{loss}\pi_t\right) + \pi_t - \gamma_{loss}\pi_{t-1} - \beta^{-1}\chi\phi_{1t-1} - \beta\gamma_{pc}\mathrm{E}_t\phi_{2t+1} + \left(\beta\gamma_{pc} + 1\right)\phi_{2t} - \phi_{2t-1} = 0, \tag{7}$$

[5] The central bank solves an intertemporal optimization problem in period t, considering the expectation channel of monetary policy, and commits itself to the computed optimal path. This is the optimal solution from a timeless perspective defined by Woodford (2003).

$$\lambda_x x_t + \phi_{1t} - \beta^{-1}\phi_{1t-1} - \kappa\phi_{2t} = 0, \tag{8}$$

$$i_t\phi_{1t} = 0, \tag{9}$$

$$\phi_{1t} \geq 0, \tag{10}$$

$$i_t \geq 0. \tag{11}$$

Equations (9), (10), and (11) are conditions for the nonnegativity constraint on nominal interest rates. The above five conditions, together with the IS curve of equation (1) and the hybrid Phillips curve of equation (2), govern the loss minimization. The optimal interest rate is determined by these conditions each period. We also need initial conditions for all variables being zero except the nominal interest rate, which takes a positive value in the steady state. When the nonnegativity constraint is not binding, i.e., $i_t > 0$, the Lagrange multiplier ϕ_{1t} becomes zero by the Kuhn-Tucker condition in equation (9), and the interest rate is determined by the conditions given by equations (1), (2), (7), and (8) with $\phi_{1t} = 0$. When the nonnegativity constraint is binding, i.e., $i_t = 0$, the interest rate is simply set to zero. The interest rate remains zero at least until the Lagrange multiplier ϕ_{1t} becomes zero.

We cannot solve this system using the standard solution method because of the nonnegativity constraint on nominal interest rates, and numerical simulations are required to obtain the path of variables under optimal monetary policy in a liquidity trap. The first-order conditions in period t given by equations (7) and (8), however, characterize qualitative features of optimal monetary policy in a liquidity trap and the economy with inflation persistence.

The first feature is that, due to the central bank's objective to minimize the change in inflation rates, i.e., $\pi_t - \gamma\pi_{t-1}$, the optimality condition includes terms to smooth inflation rates as shown in equation (7). Specifically, the expected change in inflation rates as well as the current change in inflation rates induce a strong commitment to inflation smoothing. In an economy with inflation persistence, less weight is imposed on the deviation of inflation rates from a target level than in an economy without inflation persistence.[6] Thus, agents expect more accommodative stance of the central bank against inflation and a high inflation rate accelerates along with a high expected inflation rate.

The second feature of optimal monetary policy is forward-looking terms associated with introducing inflation persistence into the model. The central bank implements monetary policy based on a forecast of future inflation rates and the output gap. There are two channels to make optimal monetary policy forward-looking. The first channel functions through the parameter γ_{loss} on the future inflation rate in equation (7). Optimal monetary policy in a model with inflation persistence should respond to the expected inflation rate. The second channel works through the parameter γ_{pc} in equation (7) on the Lagrange multiplier ϕ_{2t+1} that is related to the future output gap and a future zero interest rate condition. Note that the optimality condition includes the backward-looking variables, which induces history dependent policy as in the standard new Keynesian model. Theoretically, both forward-looking and backward-looking elements contribute to determining the optimal path of the nominal interest rates, including the optimal timing of exit from the zero interest rate.

When comparing the optimal targeting rule with that in the previous literature, the features of optimal monetary policy become evident.[7]

[6] We also see a feature of an accommodative stance of a central bank against inflation in the central bank's loss function. Assuming that the inflation rate in equilibrium is expressed in the form of AR(1) process with a persistence of ρ_π, equation (5) can be reduced to

$$\mathrm{L}_t = (\rho_\pi - \gamma)^2 \pi_{t-1}^2 + \lambda_x x_t^2.$$

For reasonable parameters, $(\rho_\pi - \gamma)^2$ is less than one and a central bank stabilizes an economy by imposing less weight on inflation rates.

[7] We can derive an optimal price-level targeting rule that exactly achieves the same optimal commitment solution as the inflation targeting rule. Defining a price-level $\widetilde{p}_t$ and a price-level target p_t^* as

$$\widetilde{p}_t \equiv p_t - \gamma p_{t-1} + \frac{\lambda_x}{\kappa} x_t,$$

$$
\begin{aligned}
&\beta\gamma_{pc}\mathrm{E}_t\phi_{1t+1} - (1+\gamma_{pc}+\beta\gamma_{pc})\,\phi_{1t} + \left(1+\beta^{-1}+\gamma_{pc}+\beta^{-1}\kappa\chi\right)\phi_{1t-1} - \beta^{-1}\phi_{1t-2} \\
= \quad & -\kappa\beta\gamma_{loss}\left(\mathrm{E}_t\pi_{t+1}-\gamma_{loss}\pi_t\right) + \kappa\left(\pi_t-\gamma_{loss}\pi_{t-1}\right) - \beta\lambda_x\gamma_{pc}\mathrm{E}_t\triangle x_{t+1} + \lambda_x\triangle x_t. \quad (12)
\end{aligned}
$$

This optimal targeting rule includes the zero interest rate condition given by ϕ_1. The optimal targeting rule is forward-looking due to inflation persistence as well as backward-looking. The change in inflation rates is directly related to optimal monetary policy. The rule reveals that the coefficient on $\pi_t - \gamma_{loss}\pi_{t-1}$ is positive, i.e., there is a negative effect on ϕ_{1t}, and the zero interest rate policy should be terminated when the inflation rate sufficiently accelerates. It, however, notes that the coefficient on $\mathrm{E}_t\pi_{t+1} - \gamma_{loss}\pi_t$ is negative, i.e., there is a positive effect on ϕ_{1t}. An acceleration of the inflation rate in the future works to keep a zero interest rate policy since a central bank has the incentive to smooth inflation rates. As a result, an acceleration of the expected inflation rate induces an acceleration of the current inflation rate, which contributes to strengthening the effect of the commitment policy and increases inflation rates.[8] Therefore, the zero interest rate policy is terminated earlier.

If the nominal interest rate does not hit the zero lower bound, ϕ_1 becomes zero and the optimal targeting rule (12) can be reduced to backward-looking as shown in Woodford

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$$\phi_{1t} \equiv \kappa\left(p_t^* - \widetilde{p}_t\right),$$

we have the following optimal price-level targeting rule.

$$p_t^* \equiv \frac{\gamma\beta}{1+\gamma\beta}\mathrm{E}_t p_{t+1}^* + \frac{1}{1+\gamma\beta}p_{t-1}^* - \frac{\gamma}{1+\gamma\beta}Q_t + \frac{1}{1+\gamma\beta}\left(\gamma+\beta^{-1}-\frac{\kappa\chi}{\beta}\right)Q_{t-1} - \frac{\beta^{-1}}{1+\gamma\beta}Q_{t-2},$$

where $Q_t \equiv (p_t^* - \widetilde{p}_t)$ for simplicity. The prominent feature of the rule is $\mathrm{E}_t p_{t+1}^*$. The price-level target should depend on the future target level of price associated with future economic conditions. When γ is zero, this rule is reduced to the one in Eggertsson and Woodford (2003b,a).

[8] We make this point clearer in terms of the level of the inflation rate in Appendix A.

(2003).[9]

$$\kappa\left(\pi_t - \gamma_{loss}\pi_{t-1}\right) + \lambda_x \triangle x_t = 0.$$

Unlike equation (12), the rule is not hybrid, implying that forward-looking terms drop from the targeting rule. The forward guidance of smoothing inflation rates weakens since there is only one term for the change in inflation rates in the case where the nominal interest rate does not hit the zero lower bound. It is a phenomenon of a liquidity trap that strengthens the forward guidance with inflation persistence.

When γ is zero, this rule collapses to the standard optimal targeting rule in the forward-looking new Keynesian model as follows:

$$\kappa\pi_t + \lambda_x \triangle x_t = 0.$$

4 Natural Rate Shocks

4.1 Basic Calibration

In this section, we numerically solve the model and characterize the optimal exit strategy from the zero interest rate policy. The baseline quarterly parameters are typical for the U.S. economy as in Table 1. We set $\chi = 6.25$, $\alpha = 0.66$, and $\kappa = 0.003$ in structural equations from Woodford (2003). Based on these structural parameters, we calculate $\lambda_x = 0.003$. The natural rate shock is stochastic with variance $\sigma_r = 0.2445$ and persistence $\rho_r = 0.8$, as in Adam and Billi (2006). The steady state real interest rate is set to be 3.5 percent annually and $\beta = 0.9913$. The model is solved numerically by the collocation method and the technical methodology to implement simulations is described in Appendix B.

Figure 1 shows optimal responses of the interest rate to natural rate shocks for differ-

[9] Another form is given by

$$\kappa\left(\mathrm{E}_t\pi_{t+1} - \gamma_{loss}\pi_t\right) + \lambda_x \mathrm{E}_t \triangle x_{t+1} = 0.$$

ent inflation inertia.[10] A central bank starts the zero interest rate policy even when the natural rate shock is still positive. This is an effect of uncertainty of shocks as pointed out in Adam and Billi (2006). Even in the presence of inflation inertia, the uncertainty of the natural rate shock requires a central bank to conduct preemptive monetary easing. The additional contribution of introducing inflation persistence is that the zero interest rate policy is terminated earlier, as inflation persistence becomes larger in response to the natural rate shocks.

4.2 Optimal Exit Policy

4.2.1 One-time Shock

We assume a simple situation where a one-time shock with a persistence of $\rho_r = 0.8$ occurs in period 0. In particular, we give a 2 percent negative natural rate shock (equivalent to 8 percent annually) to make the economy into a liquidity trap.[11] We also give a larger shock, i.e., an annual 12 percent negative natural rate shock with a persistence of 0.8.

Figure 2 shows the timing of an optimal exit from a zero interest rate in response to an annual 2 percent negative natural rate shock for different degrees of inflation inertia. Interest rates are annualized in the figure. We observe several quantitative characteristics in the impulse responses.

As a common feature, a central bank sets the nominal interest rate at zero for the first several periods to bring overshooting of inflation rates and reduce real interest rates to stimulate the economy in any case. Afterwards, the central bank increases the nominal interest rate and the inflation rate returns to zero. This outcome is consistent with Eggertsson and Woodford (2003b,a) and Jung et al. (2001, 2005) that show that the zero interest rate policy continues even after the natural rate turns positive in the case of the purely forward-looking economy, i.e., $\gamma = 0$.

[10] Note that Figure 1 does not show the whole feature of optimal monetary policy in the sense that other state variables are set at zero.

[11] For example, Jung et al. (2001, 2005) assume at least a 2 percent one-time negative shock to make the economy into a liquidity trap.

The distinct feature of optimal monetary policy is early tightening with inflation inertia increasing. As shown in Figure 2, when we assume $\gamma = 0.8$, the timing to terminate the zero interest rate policy is earlier compared to the case without inflation persistence. In an economy with inflation persistence, even in response to a negative shock, the inflation rate registers a positive number for the initial period and accelerates afterward.[12] Qualitatively, two reasons are worth being mentioned. First, inflation persistence itself accelerates inflation rates in an intrinsic way as the degree of inflation inertia increases. A high inflation rate in the past contributes to increasing inflation rates in the future. These reasons contribute to an early termination of the zero interest rate policy. Second, the outcome results from the power of forward guidance by the commitment policy. In particular, a central bank should stabilize $\pi_t - \gamma\pi_{t-1}$ in approximation rather than the inflation rate itself in an economy with inflation persistence. Based on this behavior by the central bank, private agents expect that current high inflation will induce a high expected inflation rate in the future, which accelerates inflation rates. This is consistent with the Fed's forward guidance to allow inflation rates to flexibly exceed a target level of inflation rate.

To quantitatively examine how these two elements affect the inflation dynamics and the zero interest rate policy, we show a case of $\gamma_{loss} = 0$, given that other γ are set to be 0.4 in Figure 3. In this case, the economy starts with initial deflation and inflation rates remain low, unlike the case of all $\gamma = 0.4$. This result is similar to the one of $\gamma = 0$ in Figure 2. It reveals that the commitment to stabilizing $\pi_t - \gamma_{loss}\pi_{t-1}$ accelerates inflation rates. To identify which of the two terms of the change in inflation rates in equation (12) strengthens the effect of the commitment policy, we set only γ_{loss} of $-\kappa\beta\gamma_{loss}\left(\mathrm{E}_t\pi_{t+1} - \gamma_{loss}\pi_t\right)$ in optimal monetary policy to be zero. Then, inflation rates become subdued compared to the case of all $\gamma = 0.4$ but remain high compared to the case of all $\gamma_{loss} = 0$. This implies that two terms quantitatively function as accelerators of inflation rates. We also show a case of setting γ_{pc} only in the hybrid Phillips curve to be zero in Figure 3. Even though the timing to end the zero interest rate policy does

[12] Initial inflation rates can be negative for small γ such as 0.1.

not change, a central bank sets the policy rate lower compared to the case of $\gamma = 0.4$. This is an effect of no inflation persistence in the Phillips curve.

Specifically, Figure 4(a) confirms that a zero interest rate policy is terminated earlier, as the persistence of inflation becomes larger. Figure 4(a-1) shows the time lag between a period when a zero interest rate policy is terminated, T_r, and a period when the natural rate becomes positive, T_{rn}, for different degrees of inflation inertia. It is shown that an early tightening policy becomes stronger as inflation persistence becomes larger. In response to an annual 8 percent negative shock, the timing of terminating a zero interest rate policy is earlier by 2 quarters in the case of $\gamma = 0.8$ than that in the case of $\gamma = 0$. In the case of $\gamma = 0.8$, a central bank starts to increase the interest rate in the timing when the natural rate turns to be positive, i.e., $T_r - T_{rn} = 0$. There is no history dependent easing. Furthermore, the early tightening policy becomes more evident as the size of the negative natural rate shocks becomes larger. When $\gamma = 0.8$ and there is an annual 12 percent negative shock, a central bank ends the zero interest rate policy even while the natural rate remains negative since $T_r - T_{rn} = -1$. This is called front-loaded tightening, which is in stark contrast to history dependent easing. According to inflation persistence and the sizes of shock, optimal exit policy from a liquidity trap drastically changes.

In Figure 4(a-2), we investigate the time lag between a period when a zero interest rate policy is terminated and a period when the inflation rate hits its peak, T_p, since the inflation rate is one of the key variables to decide the exit from a zero interest rate policy. Figure 4(a-2) shows that $T_r - T_p$ becomes smaller as inflation inertia becomes larger. In response to an annual 8 percent negative shock, the timing of terminating a zero interest rate policy is earlier by 3 quarters in the case of $\gamma = 0.8$ than that in the case of $\gamma = 0$ in relation to the peak inflation rate. In the case of $\gamma = 0.8$, a central bank terminates the zero interest rate policy immediately after the inflation rate hits its peak, i.e., $T_r - T_p = 1$. This result is a new finding against Eggertsson and Woodford (2003b,a) and Jung et al. (2001, 2005) that show that a zero interest rate policy continues for long periods even after the inflation rate hits its peak. This tendency remains unchanged for larger negative natural rate shocks.

4.2.2 Sequential Shock

Eggertsson and Woodford (2003b,a) assume that annual 2 percent negative shocks continue to occur for several years with a certain probability of producing a prolonged liquidity trap. In a similar vein, we assume a situation where negative natural rate shocks continue for a certain period, which is a realistic assumption to replicate a liquidity trap.

Figure 4(b) shows a case where annual 2 percent negative shocks with a persistence of 0.8 continue to occur for 10 quarters. The results are similar to those for a one-time shock. Both panels 4(b-1) and 4(b-2) confirm that history dependence becomes weaker as inflation inertia becomes larger. Inflation persistence induces a nontrivial implication for the optimal exit from the zero interest rate. The timing to terminate a zero interest rate policy is earlier by 4 quarters in the case of $\gamma = 0.8$ than that in the case of $\gamma = 0$ in relation to the natural rate of interest and the peak inflation rate. With a high degree of inflation persistence, a central bank increases its policy rate even before the natural rate returns to be positive, shown as $T_r - T_{rn} = -2$. This shows the case where optimal monetary policy implements the front-loaded tightening. Moreover, in the case of $\gamma = 0.8$, the zero interest rate policy is terminated immediately after the inflation rate hits its peak, i.e., $T_r - T_p = 1$. Even if we assume a different sequential shock, i.e., annual 4 percent negative shocks with persistence of 0.8 continue to occur for 4 quarters, we can draw the same conclusion that early tightening becomes more pronounced as inflation inertia becomes stronger.

For robustness, to examine a case with a shock for a longer period, Figure 5 shows impulse responses to annually 5 percent negative natural rate shock for 10 periods without persistence under optimal monetary policy. In this case, the timing of terminating a zero interest rate policy is the same for all inflation persistence. It, however, notes that a central bank implements less monetary easing since an interest rate increases faster after a termination of the zero interest rate policy as inflation persistence becomes stronger. For an inflation rate and the output gap, a less monetary easing induces larger expansion as shown in other simulations under optimal monetary policy.[13]

[13]Even when we assume sequential cost-push shock, we can not make a long deflation under optimal

Figure 6 changes the size and length of a negative shock. For various sizes and lengths of shocks, we observe the same timing of terminating a zero interest rate policy for all inflation persistence as shown in Figure 6(a).

5 Deflation and Cost-push Shocks

In the last section, we cannot observe deflation in simulations. During the last few years, however, inflation rates were below the target of inflation rates and are sometimes negative in many countries including the U.S. This prompts a question as to how the economy behaves with deflation. To that end, we assume a negative cost-push shock in the equation of the hybrid Phillips curve. Following Adam and Billi (2006), we assume a cost-push shock with $\sigma_\mu = 0.154$ and no persistence as well as a natural rate shock and then obtain the optimal response functions.

Figure 7 shows the impulse responses to an annual 8 percent one-time negative natural rate shock with a persistence of 0.8 and annual 2 percent negative cost-push shocks continuing for 5 quarters. The combination of the two negative shocks produces deflation for the first several periods. After deflationary periods, inflation rates rise. As inflation shows more persistence, inflation rates overshoot higher. For high inflation persistence such as $\gamma = 0.8$, the nominal interest rate quickly rises and overshoots above a steady state level to control inflation rates after the zero interest rate policy.

Even in the case where the economy starts with deflation, however, an early tightening policy is optimal. In particular, this characteristic becomes more pronounced in relation to the inflation rate as shown in Figure 8(a) when inflation inertia becomes larger. With the large degree of inflation persistence, the inflation rate hits its peak after a central bank begins to raise the policy rate, i.e., $T_r - T_p = -2$ in the case of $\gamma = 0.8$. Moreover, for an annual 12 percent one-time negative natural rate shock with a persistence of 0.8 and annual 2 percent negative cost-push shocks continuing for 5 quarters, Figure 8(a-1) shows $T_r - T_{rn} = -1$ in the case of $\gamma = 0.8$, which confirms front-loaded tightening in

monetary policy.

the conduct of optimal policy.

6 Evaluation for U.S. Monetary Policy

6.1 Calibration for Recent U.S. Economy

For parameters, we simply borrow these from a representative paper for a liquidity trap analysis, Eggertsson and Woodford (2003b) and Woodford (2003), and set $\chi = 0.5$, $\kappa = 0.02$, $\theta = 7.66$, and $\lambda_x = 0.0026$ as shown in Table 2.

Evaluating inflation persistence and a long-run natural interest rate is hard in real-time. These factors, however, are critical for the short- and medium-term monetary policy analysis. When we observe persistent high inflation rates after the pandemic, we naturally suppose high inflation persistence. Kiley (2023) focuses on the evolution of the inflation persistence in a Phillips Curve using the Bayesian approach. The paper shows that inflation persistence drastically increases by post-2019 experience and is estimated as 0.86. It requires us to set a highly persistent indexation parameter in our model. We set $\gamma = 1$ and it corresponds to about 0.5 for a coefficient on an inflation lag. Our parameter setting is still conservative to describe inflation persistence.

We set a nominal interest rate as 4 percent annually in the steady state and a discount factor is given by $\beta = 0.99$. This 4 percent is a sum of the 2 percent inflation target and 2 percent of an average natural rate of interest after 1990 following Laubach and Williams (2003).[14] In the model, we suppose that a long-run inflation rate is anchored at the 2 percent by the Fed.[15] This is consistent with Eggertsson and Woodford (2003b) that

[14] The Fed estimation result is available at `https://www.newyorkfed.org/research/policy/rstar/`.

[15] In particular, when we set $\gamma = 1$, the model and all conditions eventually do not change as shown in Woodford (2004). We may replace π_t by $\hat{\pi}_t = \pi_t - \bar{\pi}$, where $\bar{\pi}$ is a target inflation rate, and all conditions except the IS curve do not change since the inflation terms are described by the first difference of inflation rates. For the IS curve, we can equivalently transform it as

$$x_t = \mathrm{E}_t x_{t+1} - \chi \left(i_t - \mathrm{E}_t \hat{\pi}_{t+1} - \bar{\pi} - r_t^n \right).$$

assume an annual 4 percent nominal interest rate in the steady state due to $\beta = 0.99$. We show alternative cases for the natural interest rate in the steady state in the next section. Recently, the Fed estimates the natural rate of interest to decrease toward 1 percent.

In simulations, we interpret the second quarter of 2020 as the starting point since the FOMC has lowered the target range for the federal funds rate to 0 to 1/4 percent on March 15, 2020, and we observe large drops in gross domestic product and inflation rates on a quarterly base in the second quarter of 2020. In the third quarter of 2020, FOMC officially declared a commitment to allow inflation rates to exceed 2 percent on September 16, 2020, in their official statement. We define that the Fed changes a monetary policy regime and newly introduces optimal zero interest policy in the second quarter of 2020.[16]

Regarding shocks in the simulation, the pandemic induces a very large size shock, but a very short-term shock. We observe a large drop in the growth rate of gross domestic product only in the second quarter of 2020. Thus, we give one-time negative natural rate shock and one-time negative cost-push shock without shock persistence as Eggertsson and Woodford (2003b) to match simulations to the data at the second quarter of 2020 for an inflation rate and the output gap as shown in Figure 9.[17] The simulations are deterministic and we use Dynare to run simulations.[18]

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[16]We assume that the Lagrange multipliers ϕ_1 and ϕ_2 are zero before shocks, i.e., the second quarter of 2020.

[17]We assume -25.4 percent of the natural rate shock and -1.48 percent of cost-push shock at a time zero as a quarterly base. In simulations, we use the data in the first quarter of 2020 for the inflation rate as 0.325 (a deviation from the 2 percent target, annually). We use the Real Gross Domestic Product, Billions of Chained 2012 Dollars, Quarterly, Seasonally Adjusted Annual Rate for the output gap. We make trend series by one year moving average and calculate a gap from the trend series to real GDP. For inflation rates, we use the Consumer Price Index for All Urban Consumers: All Items Less Food and Energy in the U.S. City Average, Percent Change, Quarterly, Seasonally. For the Fed's policy rate, we use the Federal Funds Effective Rate, Percent, Quarterly, Not Seasonally Adjusted, Average.

[18]We extend a code by Johannes Pfeifer for optimal monetary policy in a liquid-

6.2 Optimal Monetary Policy for the U.S.

Figure 9 shows inflation rates, the output gap, and policy rates under optimal monetary policy and these U.S. data from the second quarter of 2020 to the first quarter of 2024.

We observe that the Fed's monetary policy shares several common points with optimal monetary policy. First, the zero interest rate policy continues even after inflation rates sufficiently exceed the 2 percent target. Optimal monetary policy continues the zero interest rate policy until the second quarter of 2022 and the Fed terminates it one quarter ahead. Moreover, the zero interest rate policy is terminated after the peak of inflation rates. An inflation rate hits a peak in the second quarter of 2021 and the first quarter of 2022 in the data and the simulation, respectively. Second, inflation rates sufficiently overshoot the target rate for over three years until the latest quarter, i.e., the first quarter of 2024. A recent chronic high inflation is the natural outcome of optimal monetary policy under a high inflation persistence. Third, the policy rates also overshoot the steady state level to stop high inflation rates until the first quarter of 2024.

Furthermore, several points need to be mentioned. Inflation surge under optimal monetary policy can explain about 70 percent of inflation data for 2021 and 2022 years. The average inflation rates for 2021 and 2022 years are 5.4 percent and 3.9 percent in the data and the simulation, respectively. However, inflation rates in the simulation are still lower than the data. A reason for it can be a cost-push shock after the pandemic. More importantly, another reason is that optimal monetary policy raises the policy rates faster than the Fed does even though the Fed terminates a zero interest rate policy one quarter earlier. This stronger and faster monetary policy tightening constrains inflation rates under optimal monetary policy. The remaining 30 percent can be constrained by the Fed's more aggressive monetary policy tightening after terminating the zero interest rate policy.

ity trap, `JohannesPfeifer/DSGE_mod/blob/master/Gali_2015/Gali_2015_chapter_5_commitment_ZLB.mod`. Our code is available upon your request.

6.3 Robust Analysis

High or Low Natural Interest Rate

We show how the nominal interest rate level in the steady state affects U.S. monetary policy. In particular, we show a case in which the U.S. natural interest rate returns to 3 percent as in the early 2000s or decreases to 1 percent following a recent decrease as shown by the Fed.[19] To set the nominal interest rates as 3 and 5 annually, we set $\beta = 0.9925$ and $\beta = 0.9876$, respectively.

Figure 11 shows a simulation result in the case of the 3 percent natural interest rate.[20] This result is similar to the case of the 2 percent natural interest rate. A difference from the case of the 2 percent natural interest rate is a faster policy rate rise after the zero interest rate policy. A reason for this is that the monetary easing becomes stronger for the same zero interest rate policy as the nominal interest rate in the long run becomes higher by a higher natural interest rate, as described in equation (1).

Figure 12 shows a simulation result in the case of the 1 percent natural interest rate.[21] In this case, the zero interest rate policy is terminated one quarter later in comparison to the case of the 2 percent natural interest rate. Thanks to the later termination, inflation rates are slightly higher and the average inflation rate for 2021 and 2022 years is 4.1 percent in the simulation. When we compare this case and the Fed's monetary policy, the timings to terminate the zero interest rate policy differ by two quarters. However, the monetary tightening, including the policy rate hike, seems to be the same amount and the nominal interest rates and inflation rates converge to the almost same levels in the first quarter of 2024.

[19] The Fed estimation result is available at `https://www.newyorkfed.org/research/policy/rstar/`.

[20] We assume -26.3 percent of the natural rate shock and -1.46 percent of cost-push shock at a time zero as a quarterly base.

[21] We assume -24.2 percent of the natural rate shock and -1.47 percent of cost-push shock at a time zero as a quarterly base.

Discounted Euler Equation

Del Negro et al. (2012) and McKay et al. (2016) point out that forward guidance by the commitment policy is extremely powerful in a liquidity trap in that it drastically raises the inflation rate and the output gap. We introduce the discounted Euler equation following McKay et al. (2016) as

$$x_t = \delta \mathrm{E}_t x_{t+1} - \xi\chi \left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n\right).$$

The discounted Euler equation is different from the IS curve since discounting parameters δ and ξ are multiplied by the expected output gap and the real interest rate, respectively. The effects of future real interest rates are discounted, and the forward guidance should be less powerful. The first-order condition of equations (7) and (8) are replaced by

$$-\beta\gamma_{loss}\left(\mathrm{E}_t\pi_{t+1} - \gamma_{loss}\pi_t\right) + \pi_t - \gamma_{loss}\pi_{t-1} - \beta^{-1}\xi\chi\phi_{1t-1} - \beta\gamma_{pc}\mathrm{E}_t\phi_{2t+1} + \left(\beta\gamma_{pc} + 1\right)\phi_{2t} - \phi_{2t-1} = 0,$$

$$\lambda_x x_t + \phi_{1t} - \delta\beta^{-1}\phi_{1t-1} - \kappa\phi_{2t} = 0.$$

In simulation, we set $\delta = 0.97$ and $\xi = 0.75$ following McKay et al. (2016) as shown in Table 2. Figure 13 shows a simulation result with the discounted Euler equation.[22] A clear difference from Figure 9 is two quarters later termination in the zero interest rate policy. The simulated output gap is closer to the data and it implies that the discounting parameters δ and ξ make the IS curve better fit the data.

7 Concluding Remarks

Under optimal monetary policy in a liquidity trap, the zero interest rate policy continues even after inflation rates are sufficiently accelerated over the 2 percent target. Eventually, inflation rates surge and the policy rates continue to overshoot the long-run level to stop

[22]We assume -29.2 percent of the natural rate shock and -1.4 percent of cost-push shock at a time zero as a quarterly base.

high inflation rates after the exit from the zero interest rate policy. The Fed's monetary policy in the exit strategy from a liquidity trap shares these features.

Furthermore, about 70 percent of high inflation is explained by optimal monetary policy. However, the remaining 30 percent can be constrained by the Fed's more aggressive monetary policy tightening after the zero interest rate policy.

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Table 1: Parameter Values

Parameters	Values	Explanation
β	0.9913	Discount Factor
χ	6.25	Elasticity of Output Gap to Real Interest Rate
κ	0.0244	Elasticity of Inflation to Output Gap
α	0.66	Price Stickiness
λ_x	0.003	Weight for Output Gap
i^*	0.875	Steady State Interest Rate (Quarterly)
σ_r	0.2445	Standard Deviation of Natural Rate Shock
ρ_r	0.8	Persistence of Natural Rate Shock
σ_μ	0.154	Standard Deviation of Cost-push Shock

Table 2: Calibration for U.S.

Parameters	Values	Explanation
β	0.99	Discount Factor
χ	0.5	Elasticity of Output Gap to Real Interest Rate
κ	0.02	Elasticity of Inflation to Output Gap
λ_x	0.0026	Weight for Output Gap
i^*	1.01	Steady State Interest Rate (Quarterly)
δ	0.97	Discount Factor to Output Gap
ξ	0.75	Discount Factor to Real Interest Rate

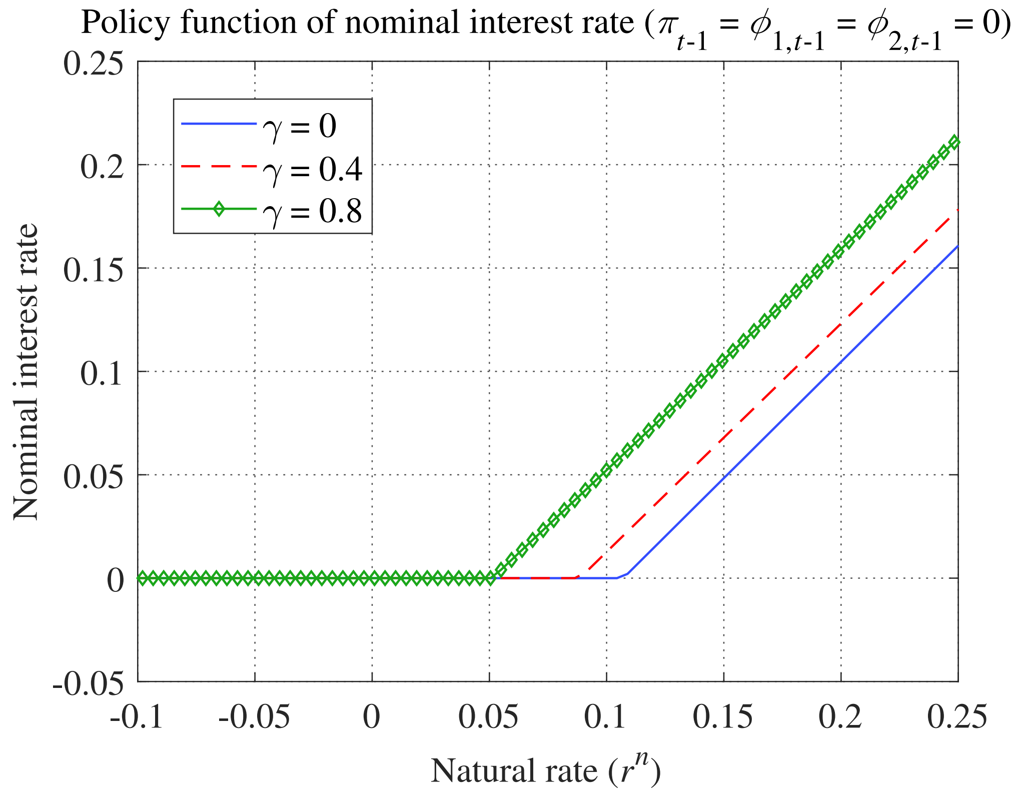

Figure 1: Optimal response of the interest rate to the natural rate shocks for different inflation inertia, where $\pi_{t-1} = \phi_{1t-1} = \phi_{2t-1} = 0$.

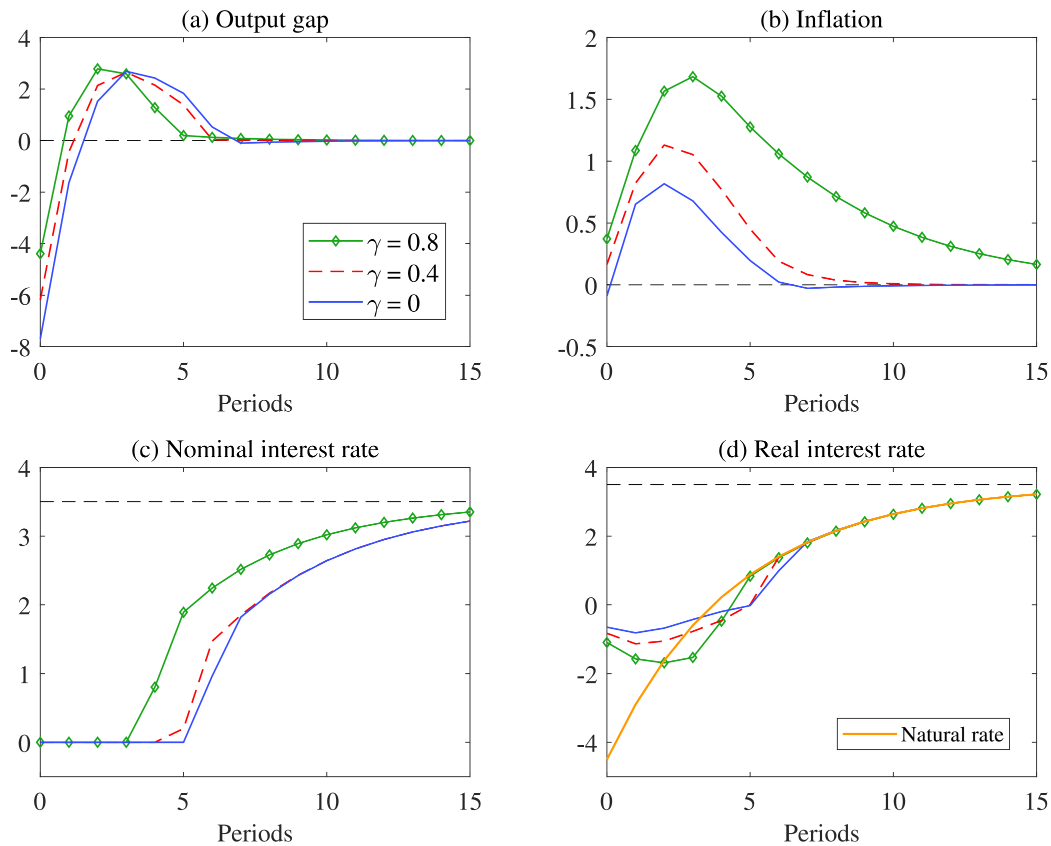

Figure 2: Impulse responses to an annual -8 percent one-time natural rate shock with a persistence 0.8.

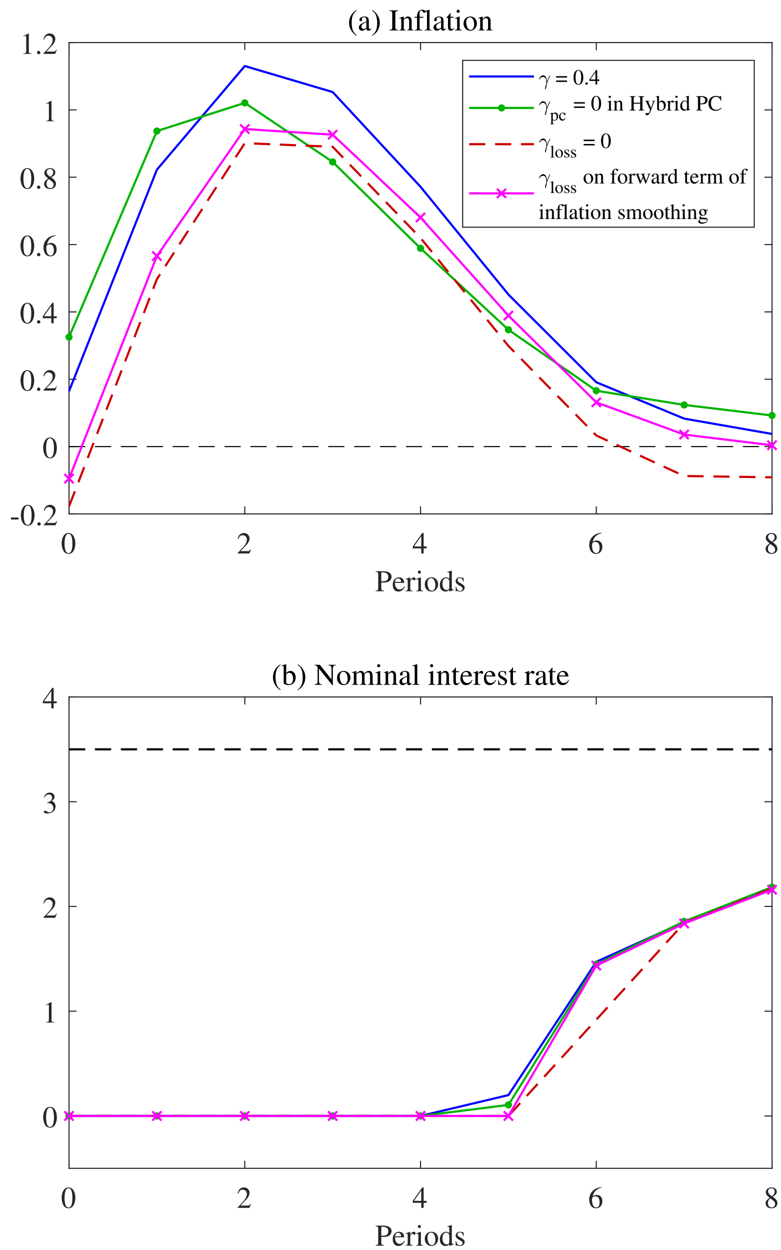

Figure 3: Impulse responses to an annual -8 percent one-time natural rate shock with a persistence of 0.8 when $\gamma = 0.4$. A solid line denotes the case of $\gamma = 0.4$. A dashed line denotes the case of $\gamma_{loss} = 0$. A line marked with circles denotes the case of $\gamma_{pc} = 0$ in a structural equation. A solid line marked with cross mark denotes the case of $\gamma_{loss} = 0$ on a forward term of inflation smoothing.

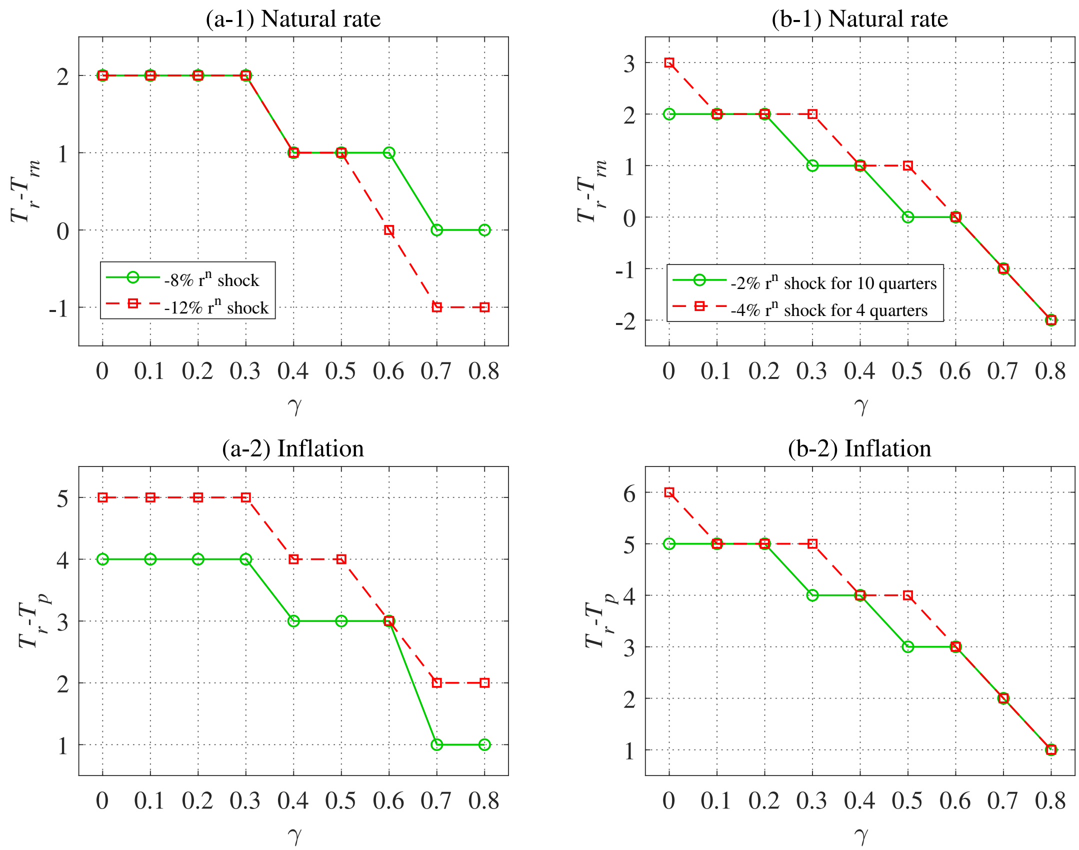

Figure 4: T_r denotes a time when the zero interest rate policy ends, T_{rn} denotes a time when the natural rate returns to zero, and T_p denotes a time when inflation hits its peak. Panels (a-1) and (a-2) denote the cases of annual -8 and -12 percent one-time natural rate shocks with persistence of 0.8. Panels (b-1) and (b-2) denote the cases of annual -2 and -4 percent natural rate shocks for 10 and 4 quarters, respectively.

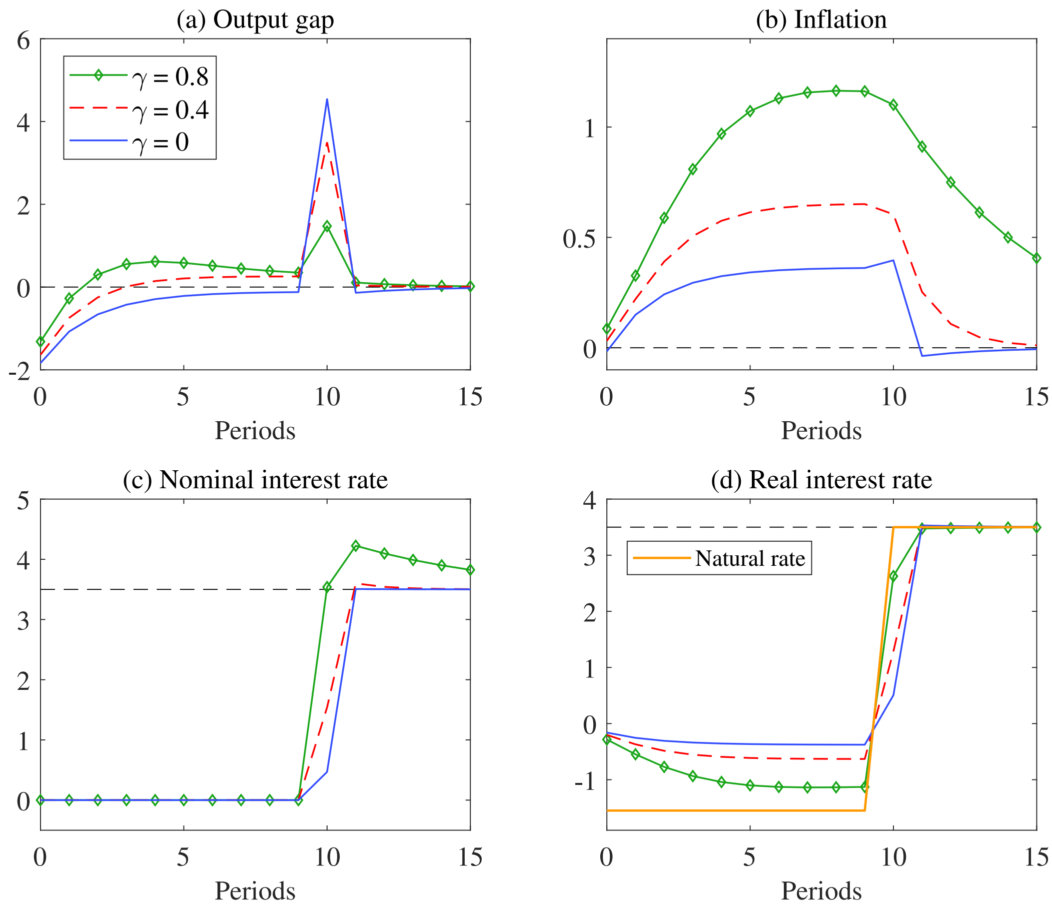

Figure 5: Impulse responses to an annual -5 percent natural rate shock without persistence for 10 quarters.

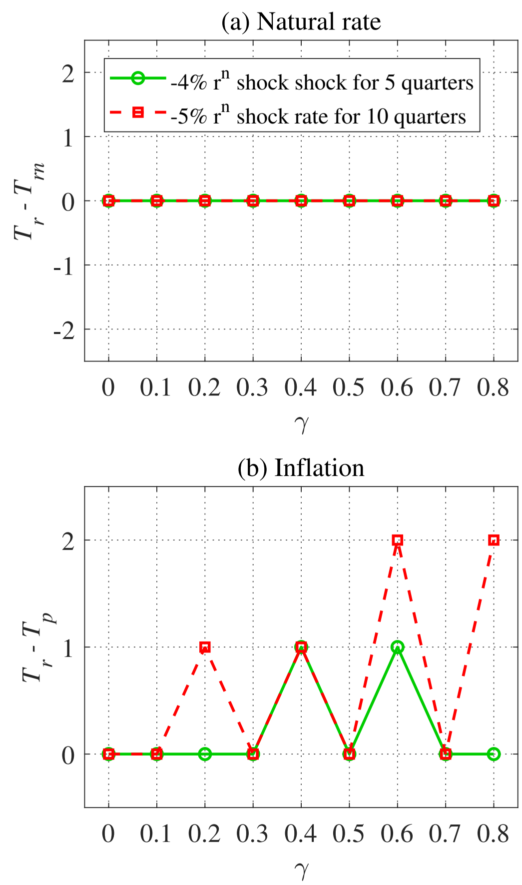

Figure 6: For T_r, T_{rn}, and T_p, see Figure 4. Panels (a) and (b) denote the cases of annual -4 and -5 percent natural rate shocks without persistence for 5 and 10 quarters.

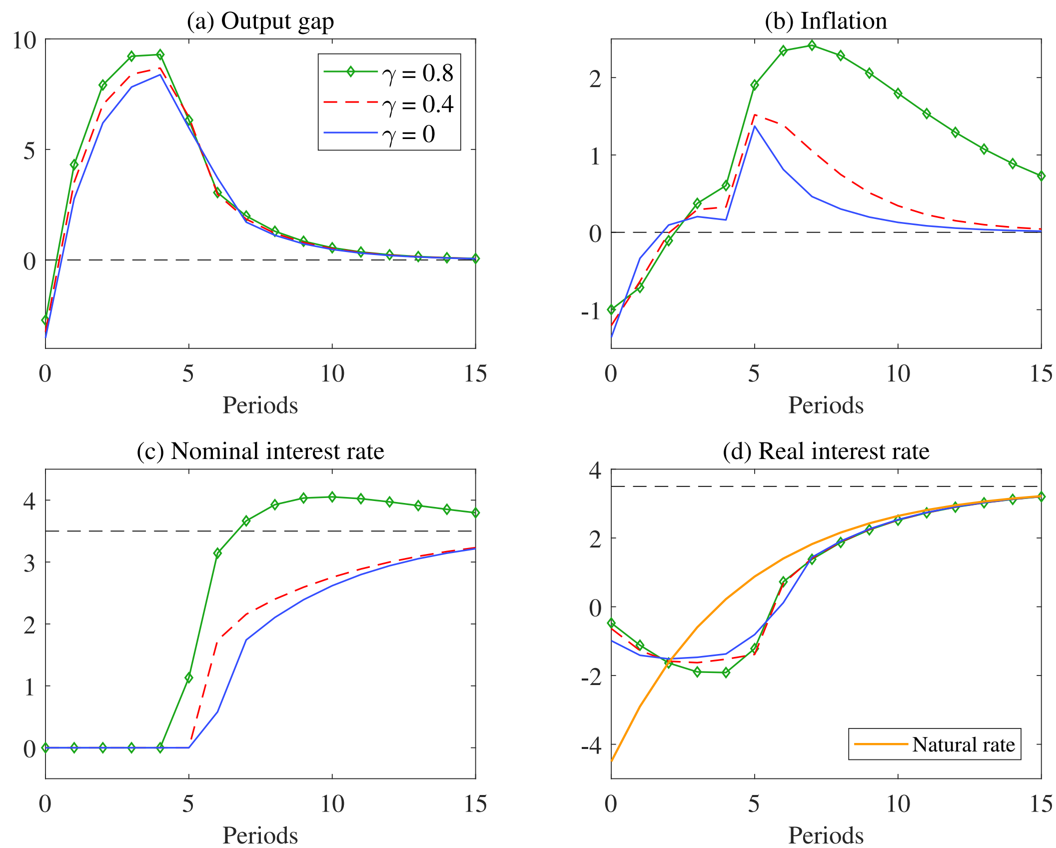

Figure 7: Impulse responses to an annual −8 percent one-time natural rate shock with a persistence of 0.8 and annual −2 percent cost-push shocks for 5 quarters.

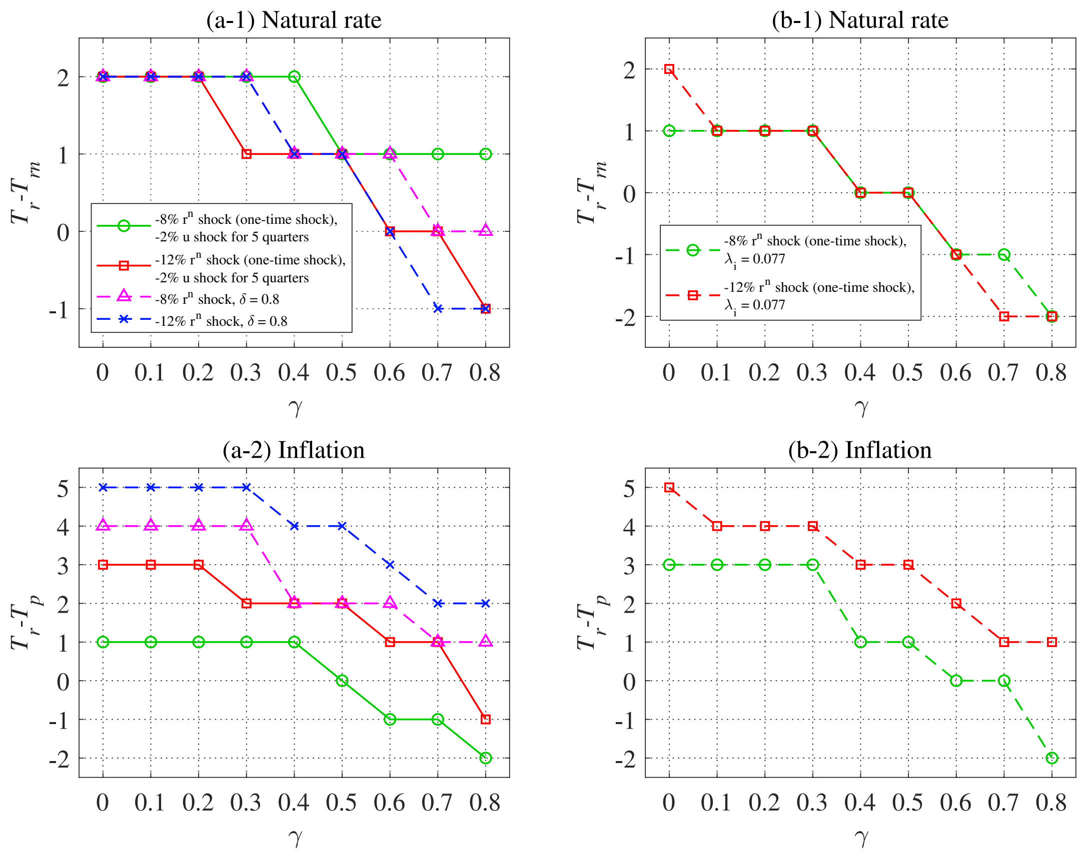

Figure 8: For T_r, T_{rn}, and T_p, see Figure 4. In Panels (a-1) and (a-2), solid lines with circles and squares denote the cases of annual -8 and -12 percent one-time natural rate shocks with persistence of 0.8 accompanied with annual -2 percent cost-push shocks for 5 quarters, respectively, and dashed lines with triangles and diamonds denote the cases of annual -8 and -12 percent one-time natural rate shocks with persistence of 0.8, respectively, when $\delta = 0.8$. In Panels (b-1) and (b-2), dashed lines with circles and squares denote the cases of annual -8 and -12 percent one-time natural rate shocks with persistence of 0.8, respectively, when $\lambda_i = 0.077$.

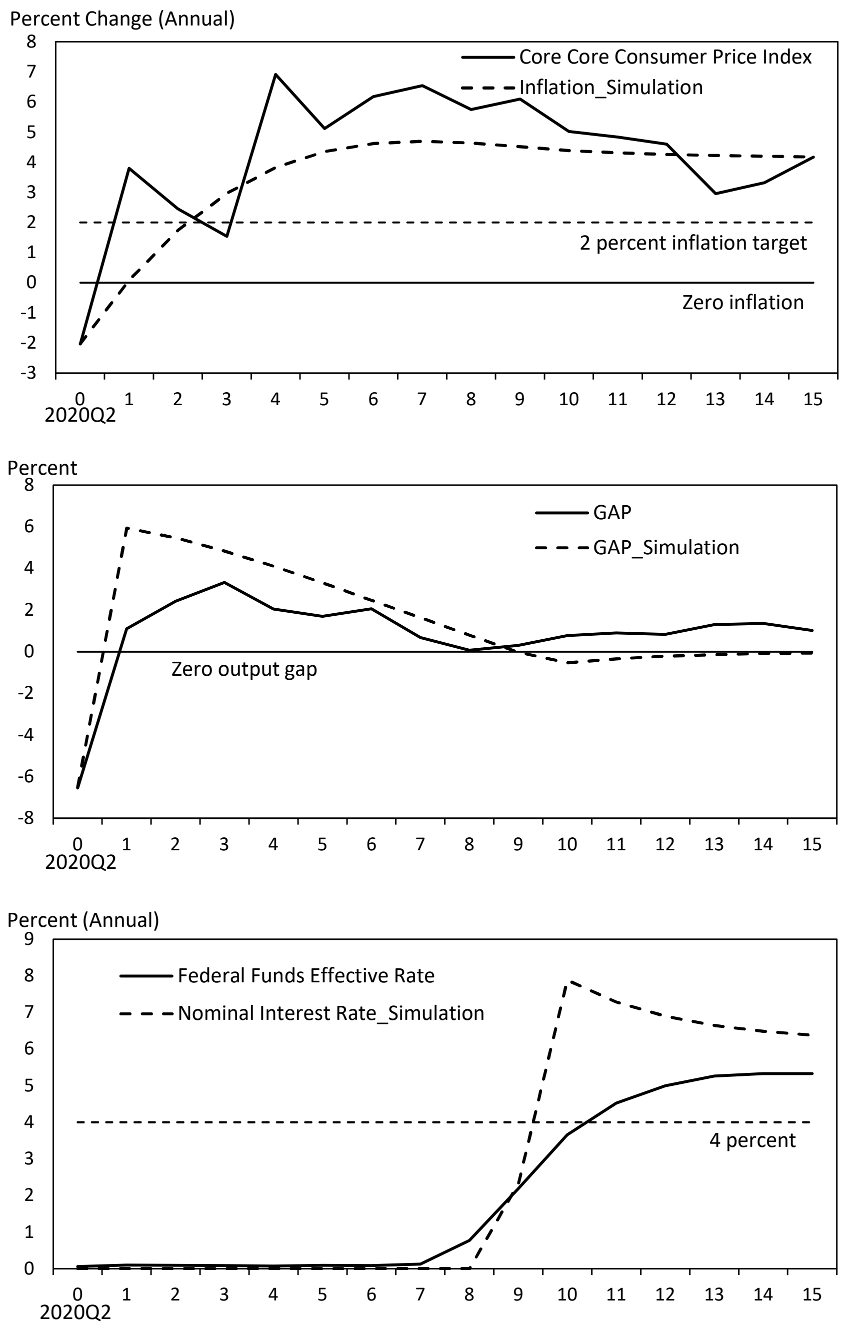

Figure 9: Simulation for U.S. Monetary Policy

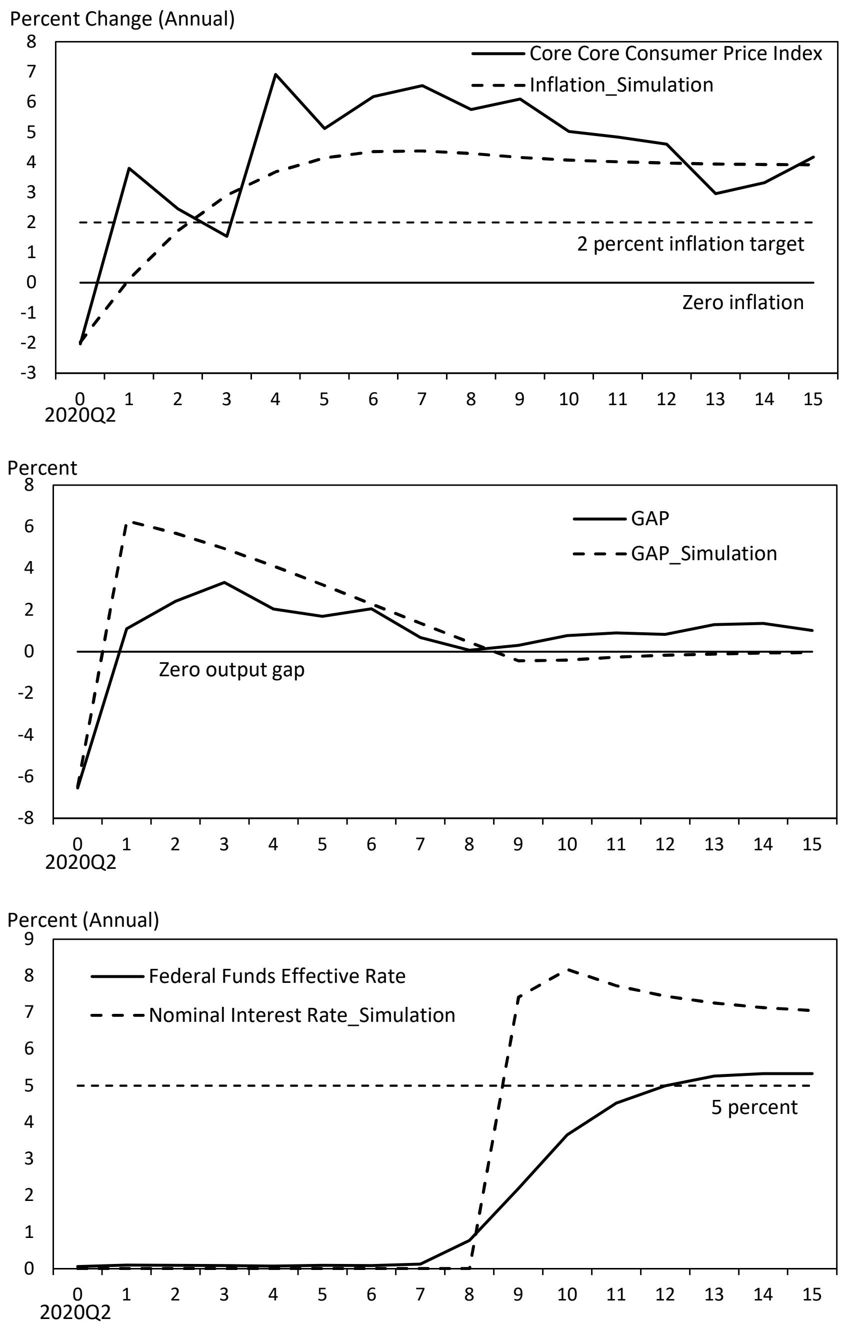

Figure 10: Simulation for U.S. Monetary Policy: High Natural Rate

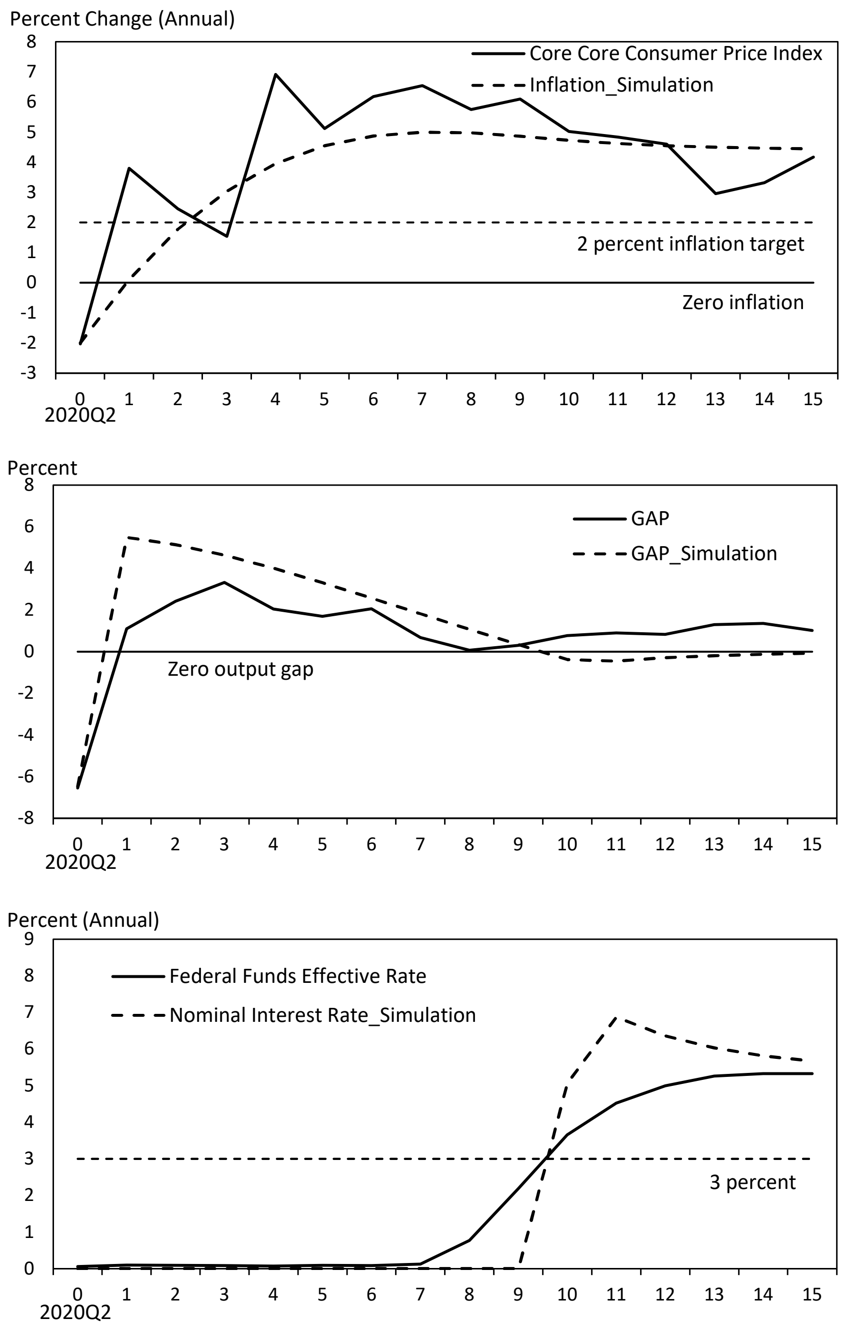

Figure 11: Simulation for U.S. Monetary Policy: Low Natural Rate

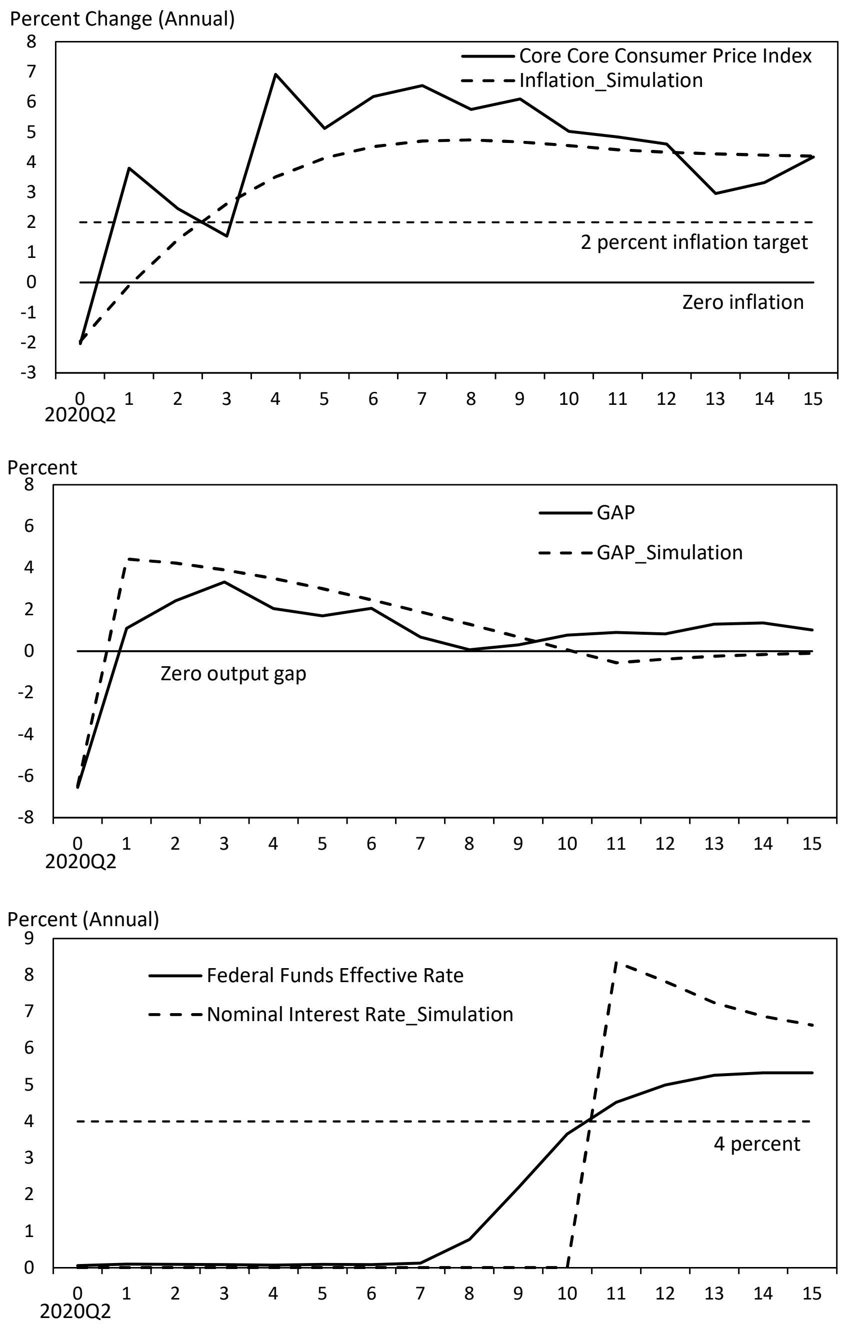

Figure 12: Simulation for U.S. Monetary Policy: Discounted Euler Equation

Appendix

A Expression in Inflation Rate

We follow the idea of Giannoni and Woodford (2003) to construct equations. We assume $\chi > 0$, $\kappa > 0$, $0 < \beta < 1$, and $0 < \gamma \leq 1$. The equation (12) becomes:[23]

$$\begin{aligned} & \beta\gamma\left(1-\Psi_1 L\right)\left(1-\Psi_2 L\right)\left(1-\Psi_3 L\right)\mathrm{E}_t\phi_{1t+1} \\ = \;& -\kappa\beta\gamma\left(\mathrm{E}_t\pi_{t+1}-\gamma\pi_t\right)+\kappa\left(\pi_t-\gamma\pi_{t-1}\right)-\beta\lambda_x\gamma\mathrm{E}_t\triangle x_{t+1}+\lambda_x\triangle x_t. \end{aligned}$$

As shown in Giannoni and Woodford (2003), we need one root with $0 < \Psi_1 < 1$ and two roots outside the unit circle to obtain a solution in the model. The two roots are either two real roots $1 < \Psi_2 \leq \Psi_3$ or a complex pair Ψ_2, Ψ_3 of which real parts are greater than one. For any γ, it is the case that

$$\begin{aligned} & -\left(1-\Psi_1 L\right)\left(1-\frac{\Psi_2+\Psi_3}{2}L\right)\phi_{1t} \\ = \;& \frac{1}{2}\left(\beta\gamma\Psi_3\right)^{-1}\mathrm{E}_t\left[\left(1-\Psi_3^{-1}L^{-1}\right)^{-1}V_t\right]+\frac{1}{2}\left(\beta\gamma\Psi_2\right)^{-1}\mathrm{E}_t\left[\left(1-\Psi_2^{-1}L^{-1}\right)^{-1}V_t\right], \end{aligned}$$

where

$$V_t \equiv -\kappa\beta\gamma\left(\mathrm{E}_t\pi_{t+1}-\gamma\pi_t\right)+\kappa\left(\pi_t-\gamma\pi_{t-1}\right)-\beta\lambda_x\gamma\mathrm{E}_t\triangle x_{t+1}+\lambda_x\triangle x_t.$$

By deconstructing these equations, we have

$$\phi_{1t}-\rho_1\phi_{1t-1}-\rho_2\triangle\phi_{1t-2} = -\frac{1}{2}\left(\beta\gamma\Psi_3\right)^{-1}m_t^I-\frac{1}{2}\left(\beta\gamma\Psi_2\right)^{-1}m_t^{II},$$

where

$$\rho_1 = \Psi_1+\frac{\Psi_2+\Psi_3}{2}-\Psi_1\frac{\Psi_2+\Psi_3}{2} > 1,$$

$$\rho_2 = \Psi_1\frac{\Psi_2+\Psi_3}{2} > 0,$$

$$m_t^I = \mathrm{E}_t\left[\left(1-\Psi_3^{-1}L^{-1}\right)^{-1}V_t\right] = \kappa\sum_{i=-1}^{\infty}\alpha_{\pi,i}^I\mathrm{E}_t\pi_{t+i}+\lambda_x\sum_{i=-1}^{\infty}\alpha_{x,i}^I\mathrm{E}_t x_{t+i},$$

[23] Thus, $\beta\gamma = \left(\beta\Psi_1\Psi_2\Psi_3\right)^{-1}$, $\beta\gamma\left(\Psi_1+\Psi_2+\Psi_3\right) = 1+\gamma+\beta\gamma$, and $\beta\gamma\Psi_1 = 1+\gamma-\beta\gamma\left(\Psi_2+\Psi_3-1\right)$.

$$m_t^{II} = \mathrm{E}_t\left[\left(1-\Psi_2^{-1}L^{-1}\right)^{-1}V_t\right] = \kappa\sum_{i=-1}^{\infty}\alpha_{\pi,i}^{II}\mathrm{E}_t\pi_{t+i} + \lambda_x\sum_{i=-1}^{\infty}\alpha_{x,i}^{II}\mathrm{E}_t x_{t+i},$$

$$\alpha_{\pi,-1}^{I} = -\gamma,$$

$$\alpha_{\pi,0}^{I} = 1+\beta\gamma^2-\gamma\Psi_3^{-1},$$

$$\alpha_{\pi,i}^{I} = -\gamma\beta\Psi_3^{-i+1}+\Psi_3^{-i}\left(1+\beta\gamma^2\right)-\Psi_3^{-i-1}\gamma,\ i=1,2,3,...,$$

$$\alpha_{x,-1}^{I} = -1,$$

$$\alpha_{x,0}^{I} = 1+\beta\gamma-\Psi_3^{-1},$$

$$\alpha_{x,i}^{I} = -\gamma\beta\Psi_3^{-i+1}+\Psi_3^{-i}\left(1+\beta\gamma\right)-\Psi_3^{-i-1},\ i=1,2,3,...,$$

$$\alpha_{\pi,-1}^{II} = -\gamma,$$

$$\alpha_{\pi,0}^{II} = 1+\beta\gamma^2-\gamma\Psi_2^{-1},$$

$$\alpha_{\pi,i}^{II} = -\gamma\beta\Psi_2^{-i+1}+\Psi_2^{-i}\left(1+\beta\gamma^2\right)-\Psi_2^{-i-1}\gamma,\ i=1,2,3,...,$$

$$\alpha_{x,-1}^{II} = -1,$$

$$\alpha_{x,0}^{II} = 1+\beta\gamma-\Psi_2^{-1},$$

$$\alpha_{x,i}^{II} = -\gamma\beta\Psi_2^{-i+1}+\Psi_2^{-i}\left(1+\beta\gamma\right)-\Psi_2^{-i-1},\ i=1,2,3,....$$

Finally, we rearrange these equations as:

$$\begin{aligned}&\phi_{1t}-\rho_1\phi_{1t-1}-\rho_2\triangle\phi_{1t-2}\\ =\ &\mathrm{E}_t\sum_{i=0}^{\infty}\alpha_{\pi,i}\pi_{t+i}+\mathrm{E}_t\sum_{i=0}^{\infty}\alpha_{x,i}x_{t+i}+\alpha_{\pi,-1}\pi_{t-1}+\alpha_{x,-1}x_{t-1},\end{aligned}$$

where

$$\alpha_{\pi,-1} = \frac{\kappa}{\beta}\frac{\Psi_2^{-1}+\Psi_3^{-1}}{2},$$

$$\alpha_{x,-1} = \frac{\lambda_x}{\beta\gamma}\frac{\Psi_2^{-1}+\Psi_3^{-1}}{2},$$

$$\alpha_{\pi,i} = -\frac{\kappa}{2\beta\gamma}\left(\Psi_3^{-1}\alpha_{\pi,i}^{I}+\Psi_2^{-1}\alpha_{\pi,i}^{II}\right),$$

$$\alpha_{x,i} = -\frac{\lambda_x}{2\beta\gamma}\left(\Psi_3^{-1}\alpha_{x,i}^{I}+\Psi_2^{-1}\alpha_{x,i}^{II}\right).$$

In particular, for a large γ, coefficients of $\alpha_{\pi,i}$ and $\alpha_{x,i}$ are positives for small i such as -1, 1, 2, and 3. For the parameters in Table 1, when $\gamma = 0.1$ and $\gamma = 0.8$, ($\alpha_{\pi,-1}$, $\alpha_{\pi,0}$, $\alpha_{\pi,1}$, $\alpha_{\pi,2}$, $\alpha_{\pi,3}$) is (0.009, -0.087, -0.041, -0.027, -0.018) and (0.015, -0.023, 0.002, 0.002, 0.002), respectively. When $\gamma = 0.1$ and $\gamma = 0.8$, ($\alpha_{x,-1}$, $\alpha_{x,0}$, $\alpha_{x,1}$, $\alpha_{x,2}$, $\alpha_{x,3}$) is (0.011, -0.006, -0.002, -0.001, -0.001) and (0.002, -0.003, 0.0001, 0.0002, 0.0002), respectively.

B Numerical Algorithm

We solve the central bank's optimization problem by calculating the solution for equations (1) to (3) and equations (7) to (11). Since the zero lower bound (ZLB) introduces nonlinearity in the model, we employ a numerical technique which approximates expected variables.

First of all, we specify the grids for four state variables, r_t^n, ϕ_{1t-1}, ϕ_{2t-1}, and π_{t-1}. Let $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\mathbf{S}_4$ denote the vector of grids for r_t^n, ϕ_{1t-1}, ϕ_{2t-1}, and π_{t-1}, respectively. A tensor of these grid vectors, defined as $\mathbf{S} \equiv \mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{S}_3 \otimes \mathbf{S}_4$, determines the combination of all grids. The size of $\mathbf{S}$ is $N = n_1 \times n_2 \times n_3 \times n_4 = 25000$. As for $\mathbf{S}_1$, we put relatively larger number of grids near the kink point stemming from the ZLB with the aim of mitigating the expected approximation error. The *p.d.f.* for the natural interest rate is discretized by Gaussian Quadrature.

Notice that we can rewrite the complementarity conditions regarding the ZLB, equations (9) to (11), as

$$\min(\max(\chi\phi_{1t}, -i_t), \infty) = 0. \tag{13}$$

In order to employ an algorithmic solution that is designed basically for differentiable functions, we approximate equation (13) by a semismooth function, so called Fischer's equation:

$$\psi^-(\psi^+(\chi\phi_{1t}, -i_t), \infty) = 0,$$

where $\psi^\pm(u, v) = u + v \pm \sqrt{u^2 + v^2}$ (c.f., Miranda and Fackler (2004)).

Let $\mathbf{h}_t \equiv (x_t, \pi_t, \phi_{2t})$ denote the vector of forward-looking variables at time t. We need to obtain $\mathbf{h}_t$, i_t, and ϕ_{1t} by solving the central bank's optimization problem, taking state variables as given. In order to calculate the expectations terms, we approximate the time-invariant function for forward-looking variables, $\mathbf{h}$, by a collocation method. Our solution procedure is summarized as follows:

1. Given a particular set of grids for state variables, denoted by $\mathbf{S}^j$, and the initial guess of the functional form for $\mathbf{h}(\mathbf{S}^j)$, denoted by $\mathbf{h}^0(\mathbf{S}^j)$, compute $\mathbf{h}^1(\mathbf{S}^j)$, i_t, and $\phi_{1,t}$ as a solution for equations (1) to (3) and equations (7) to (11). A cubic-spline function is used to interpolate $\mathbf{h}(\mathbf{S}^\mathbf{j})$.

2. Repeat step 1 for all $j = 1, \ldots N$.

3. Stop if $\|\mathbf{h}^1 - \mathbf{h}^0\|_\infty / \|\mathbf{h}^0\|_\infty < 1.5 \times 10^{-6}$. Otherwise, update the initial functional form as $\mathbf{h}^0 \equiv \mathbf{h}^1$ and go to step 1.

Impulse responses are derived using Matlab routine fsolve with the obtained policy function as a given. Euler residuals from first order conditions are order of 10^{-3}, which is concentrated mostly around the zero lower bound. Computation time is 8 hours for each γ. The software is Matlab, CPU is Core i7 with 2.90GHz, and Memory is 16GB.